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β -EXPANSIONS AND SYMBOLIC DYNAMICS

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Abstract. Given a real number $\beta > 1$, it is easy to expand $x \in [0, 1]$ in a “canonical” way into a series of negative powers of β with integer coefficients less than or equal to $[\beta]$. The form of the set of such “ β -expansions”—which is a matter of symbolic dynamics or language theory—varies according to arithmetical properties of β , but there are many common features. Known facts (which are few) and open questions (much more numerous) are reviewed in this paper.

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Introduction

The most interesting feature of the theory of β -expansions is that it creates (or rather should eventually create) a link between symbolic dynamics and a part of number theory: e.g., for what class of numbers has the β -shift such and such properties (or the reverse)? The idea of systematically exploring that field is due to Anne Bertrand-Mathis, to whom I am indebted for most that I know about it. The Appendix contains matters which were found out at a meeting in Marseilles, at the end of my talk, by J.-P. Allouche, A. Bertrand-Mathis and C. Mauduit. It was mostly written down by J.-P. Allouche.

This paper is a strongly modified version of a previous one (in French) which is to appear in the Proceedings of the 1st Franco-Chilean Meeting on Applied Mathematics.

1. Definitions

1.1. Languages

Let the *alphabet* A be a finite set.

Definition. A *language* on A is a set of *words* on, or finite sequences of elements of, A .

Definition. A *code* on A is a language X such that, for any equality

$$x_1 x_2 \dots x_k = y_1 y_2 \dots y_n,$$

with $x_i, y_j \in X$, one has $n = k$, $x_i = y_i$.

A particular class is that of *prefix codes*: a prefix code is a language such that no word in it can be the beginning of another. It is easily checked that it is a code.

1.2. Symbolic dynamical systems

$A^{\mathbb{Z}}$ is endowed with the usual topology, and σ is the *shift* on $A^{\mathbb{Z}}$; σ is a continuous, 1-1 mapping.

Definition. A *symbolic dynamical system* (in short *subshift*) is a closed σ -invariant subset of $A^{\mathbb{Z}}$.

1.1. Proposition. A *subshift* S is completely defined by the set $L(S)$ of allowed words: all those that appear in the sequence of coordinates of its elements.

An important property of certain subshifts is the following.

Definition. A subshift S is *transitive* if

$$\forall u, v \in L(S), \exists w \in L(S) \text{ such that } uwv \in L(S).$$

Definition. A *subshift of finite type* S is a subshift defined by forbidding a finite set of words.

It is always possible to suppose that only words of length 2 are forbidden, by changing the alphabet. Such a subshift is defined by a graph with vertices in A , and with edges corresponding to allowable 2-letter words.

1.2. Example. Forbidding the word bb defines a subshift of finite type on alphabet $\{a, b\}$, with graph

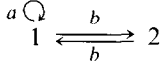


Definition. A *sofic system* is a subshift S such that the language $L(S)$ is *regular*, that is, recognizable by a finite automaton: let G be a finite graph with an arbitrary

set of vertices and with edges labelled by letters of A , with at most one edge with given label from one vertex to another. Accepted words are those that can be spelled by paths in the graph.

An equivalent property is that $L(S)$ is *rational* (definition in [12]).

1.3. Example. Sequences of $\{a, b\}^{\mathbb{Z}}$ containing only strings of b 's with even length form a sofic system which is not of finite type:



is a finite automaton recognizing the subshift that cannot be defined by forbidding any finite set of words.

All subshifts of finite type are sofic: their graph is a particular automaton which recognizes them. For any set of words Y , define Y^{∞} as the subset of $A^{\mathbb{Z}}$ of all sequences which can be broken in at least one way into words belonging to Y .

Definition. A *coded system* is a subshift S for which there exists a (non-unique) language Y such that S is the closure of Y^{∞} .

For S coded, there always exists a code X such that $X^{\infty} = S$ [8]. All coded systems are transitive. All transitive sofic systems are coded, but the converse is false, as shown in the following example.

1.4. Example. Here is a nonsofic coded system: for

$$Y = \{a^n b^n \mid n \in \mathbb{N}\},$$

the closure of Y^{∞} , S_Y , is the subset of $A^{\mathbb{Z}}$ in which any finite string of a 's with length n is followed by a string of b 's with length exactly n . By definition it is a coded system. Any automaton recognizing S_Y has infinitely many vertices.

Two families of coded systems may play a role in the field of β -shifts:

Definition (cf. [4]). A subshift S has the *specification property* (shorter is specified) if

$$\exists k: \quad \forall u, v \in L(S), \exists w \in L(S), \quad w = |k| \text{ such that } u w v \in L(S).$$

Definition (cf. [8]). A subshift S is *synchronizing* if there exists a word u in $L(S)$ such that if $vu \in L(S)$, then $vuw \in L(S)$ iff $uw \in L(S)$.

1.3. Classes of numbers

The *characteristic polynomial* of an algebraic number is the polynomial on \mathbb{Z} with least degree of which that number is a root. Its *conjugates* are the other roots of its characteristic polynomial.

Definition. An *algebraic integer* is an algebraic number the characteristic polynomial of which has a term of highest degree with coefficient 1.

1.5. Examples. A noninteger rational number p/q cannot be an algebraic integer (characteristic polynomial $qX - p$); $\sqrt{2}$ is an algebraic integer (characteristic polynomial $X^2 - 2$).

Definition. A *Perron number* is an algebraic integer greater than 1 all conjugates of which have absolute value less than that number.

1.6. Examples. $\sqrt{2}$ is not a Perron number, because its conjugate has the same absolute value. But $\frac{1}{2}(5 + \sqrt{5})$ is a Perron number (characteristic polynomial $X^2 - 5X + 5$).

Definition. A *Pisot* (respectively *Salem*) *number* is a Perron number all conjugates of which have absolute value strictly less than 1 (respectively less than or equal to 1, with at least one conjugate with absolute value 1).

1.7. Examples. $\frac{1}{2}(5 + \sqrt{5})$ is neither Pisot nor Salem (its conjugate is greater than 1). All integers are Pisot, as well as $\frac{1}{2}(1 + \sqrt{5})$ (characteristic polynomial $X^2 - X - 1$).

2. β -Expansions and β -shifts

For $t \in \mathbb{R}_+$, let $[t]$ be the integer part of t .

Suppose we wish to expand a real number $x < 1$ into a series of negative powers of a real number $\beta > 1$, that is, to write it down as a sum

$$x = \sum_{n=0}^{\infty} u_n(x, \beta) \cdot \beta^{-n}.$$

Of course there exist infinitely many ways of choosing a sequence $(u_n(x, \beta), n \in \mathbb{N})$ such that the formula holds, even if u_n must be an integer belonging to $A = \{0, [\beta]\}$. We shall do it exactly the same way as in the case where β is an integer.

Let $f_\beta: [0, 1] \rightarrow [0, 1]$ be the mapping

$$f_\beta(x) = \beta x - [\beta x] = \beta x \bmod 1.$$

Definition. The *expansion of $x \in [0, 1]$ in basis β* or, more briefly, the *β -expansion of x* is a sequence of integers of $\{0, 1, \dots, [\beta]\}$:

$$d(x, \beta) = (d_n(x, \beta), n > 0)$$

with

$$d_n(x, \beta) = [\beta f_\beta^{n-1}(x)], \quad n > 0.$$

($d_n(x, \beta)$ will be denoted d_n whenever there is no risk of mixup.)

Remark. As an integer, 1 cannot be the image of a real number by the mapping f_β .

Is there a way to tell whether any given infinite sequence on A is a β -expansion? Whenever β is an integer, they all are, except those that terminate with an infinite sequence of β -1's. Without this hypothesis, things are more complicated.

Let us call D_β the set of β -expansions of real numbers in $[0, 1[$ (hence, leaving out $d(1, \beta)$). It is a part of the set $\{0, 1, \dots, [\beta]\}^\mathbb{N}$, which may be endowed with the *lexicographical* order, the product topology and the (one-sided) shift.

2.1. Proposition. (1) $d(\cdot, \beta)$ is a strictly increasing mapping from $[0, 1]$ to $\{0, 1, \dots, [\beta]\}^\mathbb{Z}$; hence it is 1-1 between $[0, 1]$ and D_β .

(2) It is right-continuous but not continuous; for fixed β , d^{-1} is a continuous map from D_β to $[0, 1[$.

(3) It takes f_β to σ , that is to say, the following diagram is commutative:

$$\begin{array}{ccc} [0, 1[& \xrightarrow{f_\beta} & [0, 1[\\ d \downarrow & & \downarrow d \\ D_\beta & \xrightarrow{\sigma} & D_\beta \end{array}$$

Proof. The same as with an integer β . \square

Corollary. For any $x \in [0, 1[$, any $n \in \mathbb{N}$, $\sigma^n(d(x, \beta)) < d(1, \beta)$.

Proof. It suffices to remark that $\sigma^n(d(x, \beta)) = d(f^n(x, \beta))$ and use the fact that d is increasing. \square

Now we have got a necessary condition for a sequence in X to belong to D_β ; it is sufficient in most cases.

2.2. Examples. (1) $\beta = \frac{1}{2}(1 + \sqrt{5})$: $d(1, \beta) = 110000000 \dots$. Whenever the expansion of 1 ends with an infinite string of 0's, we shall say it is *finite*.

(2) $\beta = \frac{1}{2}(3 + \sqrt{5})$: $d(1, \beta) = 2111111111 \dots$. In equivalent cases, we shall say the expansion of 1 is *ultimately periodic*.

(3) $\beta = \frac{3}{2}$: $d(1, \beta) = 101000001 \dots$. We shall prove that it cannot be ultimately periodic for that value of β .

The set D_β is characterized thus.

2.3. Proposition (Renyi [17], Schmidt [18]). (1) When the β -expansion of 1 is not finite, the condition

$$\sigma^n(s) < d(1, \beta) \quad \forall n \in \mathbb{N}$$

is necessary and sufficient for the sequence $s \in X$ to belong to D_β .

(2) If $d(1, \beta) = d_0 d_1 \dots d_k 0000000 \dots$, then s belongs to D_β if and only if $\forall n \in \mathbb{N}$, $\sigma^n(s)$ is lexicographically less than the periodic sequence

$$d^*(\beta) = d_0 d_1 \dots d_{k-1} (d_k - 1) d_0 d_1 \dots d_{k-1} (d_k - 1) d_0 d_1 \dots d_{k-1} (d_k - 1) \dots$$

Let us point out that $d^*(1, \beta)$ is a sort of incorrect β -expansion of 1, just like $0.9999999999 \dots$ in basis 10.

Remark. Notice that if $\beta \neq \beta'$ then $d(1, \beta) \neq d(1, \beta')$: that is, it is a consequence of the equation in β

$$1 = \sum_{n=0}^{\infty} s_n \beta^{n-1} \quad (s \in X)$$

having a unique positive solution.

Proposition 2.3 tells us that D_β is the set of all one-sided sequences that are lexicographically strictly less than $d(1, \beta)$. Hence it is also the set of all sequences that split into words belonging to the prefix code

$$Y_\beta = \{d_0 d_1 \dots d_{n-1} b \mid b < d_n, n \in \mathbb{N}\}.$$

D_β is not properly the set Y_β^x , but its one-sided version. It is natural now to introduce the subshift S_β , which is the closure of the extension of D_β to two-sided sequences. Notice that

$$S_\beta = \{s \in \mathbb{Z} \mid \forall m < n, s_m s_{m+1} \dots s_n \leq d(1, \beta)\}$$

whenever $d(1, \beta)$ is not finite (similar formula with $d^*(1, \beta)$ when it is).

Remark. S_β is the coded system generated by the set Y_β . Hence *all β -shifts are coded*.

Most properties interesting from the point of view of number theory concern D_β , but S_β is a subshift and that allows us to use some powerful theorems of symbolic dynamics.

2.4. Proposition. *A sequence $s \in \{0, 1, \dots, [\beta]\}^*$ is the β -expansion of 1 for a certain β iff $\sigma^n s < s$, $n > 0$. Then β is unique.*

Proof (outline). β must be the unique solution of the equation

$$1 = \sum_{n \geq 0} s_n \beta^{-n};$$

owing to the hypothesis, it is easy to check that the β -expansion of 1 is equal to s . \square

3. Common properties of β -shifts

In this section and the next, some terms of symbolic dynamics or language theory have not been defined previously. Definitions can be found in the papers referred to.

- (1) *All β -shifts are coded.* See above.
- (2) *The topological entropy of a β -shift is equal to $\text{Log } \beta$* [16, 17].
- (3) *The set Y_β is an exhaustive code* [7].
- (4) *The ζ -function of S_β is equal to $1/(1 - F(z, Y_\beta))$, where $F(z, Y_\beta)$ is the generating function of Y_β :*

$$F(z, Y_\beta) = \sum_{n \in \mathbb{N}} z^n \text{card}(Y_\beta \cap A^n).$$

The calculation was done in [19] for D_β and completed for S_β in [7].

- (5) *β -Shifts are intrinsically ergodic.* First proved in [13], shorter proof in [6]. This result goes over to the dynamical system $([0, 1[, f_\beta)$.

Question. Does there exist a topological property, common to all β -shifts and stable under homomorphisms, implying intrinsic ergodicity? (For instance, transitive sofic systems are intrinsically ergodic; but most β -shifts are not sofic.)

- (6) *Generic points can be constructed for the measure of maximal entropy* [3].

4. Classes of numbers and classes of subshifts

Perhaps in the future it will be possible to give a classification of real numbers according to the ergodic properties of their β -shifts. We are yet far from that aim, but one thing is already sure: such a classification would be quite different from the usual, algebraic one. For instance, ergodically speaking, noninteger rational numbers are far removed from integers. It is not the *degree* of the characteristic polynomial that matters, it is its form.

4.1. Properties of $d(1, \beta)$

The link between topological properties of $d(1, \beta)$ —in fact those of the orbit of 1 under f_β —and ergodic properties of S_β is completely known. So questions that are still open concern the relationship between the former and the classes of numbers to which β belongs.

Class 1: $d(1, \beta)$ is finite

An obvious equivalent is that β satisfies a polynomial equation

$$\beta^{n+1} = d_0 \beta^n + \dots + d_n,$$

with $d_0 d_1 \dots d_n 000000 \dots$ satisfying the hypothesis of Proposition 2.4 (which implies $d(1, \beta) = d_0 d_1 \dots d_n 000000 \dots$).

4.1. Proposition (Parry [16]). S_β is a subshift of finite type iff $d(1, \beta)$ is finite.

Some Pisot numbers (e.g., $\frac{1}{2}(1+\sqrt{5})$) belong to that class, others do not (e.g., $\frac{1}{2}(3+\sqrt{5})$).

Class 2: $d(1, \beta)$ is ultimately periodic but not finite

4.2. Proposition (Bertrand-Mathis [5]). S_β is sofic iff $d(1, \beta)$ is ultimately periodic.

It is easy to prove that for β in that class S_β is *almost never of finite type* [9].

4.3. Proposition (Parry [16]). If β is a Pisot number, then S_β is sofic.

4.4. Proposition (Lind [14], Denker, Grillenberger and Sigmund [11]). If S_β is sofic, then β is a Perron number.

There exist numbers which are neither Pisot nor Salem such that S_β is sofic or even finite type: for instance, the positive solutions of equations $\beta^{n+1} = \beta^n + 1$ (n big enough) and $\beta^4 = 3\beta^3 + 2\beta^2 + 3$ cannot be Pisot nor Salem (there exists a conjugate with absolute value > 1), but the equations yield the expansion of 1, which is finite.

It is easy to prove that if β is a Perron number with a *real* conjugate greater than 1, then $d(1, \beta)$ cannot be ultimately periodic. Hence the converse to Proposition 4.4 is false.

When β is a noninteger rational number, hence not an algebraic integer, S_β is never sofic.

Question. Characterize the set of real numbers β for which S_β is sofic. Does it contain all Salem numbers?

Class 3: $d(1, \beta)$ contains bounded strings of 0's, but is not ultimately periodic

4.5. Proposition (Bertrand-Mathis [5]). S_β is specified iff there exists an $n \in \mathbb{N}$ such that all strings of 0's in $d(1, \beta)$ have length less than n .

Apart from what is stated in the Appendix, only one thing is known: such numbers exist.

4.6. Example. Let $s_n = 2$ whenever $n = 2^p$, $p \in \mathbb{N}$, $s_n = 1$ for other values of n . The sequence s satisfies the hypothesis of Proposition 2.4: it is equal to $d(1, \beta)$ for some β , and S_β is specified.

Class 4: $d(1, \beta)$ does not contain some word of $L(S_\beta)$, but contains strings of 0's with unbounded length

4.7. Proposition (Bertrand-Mathis [5]). S_β is synchronizing iff $d(1, \beta)$ does not contain some word of $L(S_\beta)$ in the sequence of its coordinates.

Same remark as for Class 3.

4.8. Example. It is obtained by replacing 2's by 1's and 1's by 0's in Example 4.6; the sequence has no longer bounded strings of 0's, but strings of exactly $2^p + 1$ zeros between two 1's, which form words belonging to $L(S_\beta)$, are not to be found in it.

Class 5: $d(1, \beta)$ contains all words of $L(S_\beta)$

Probably the widest class. It is possible, but tedious, to construct sequences satisfying the hypothesis of Proposition 2.4 and belonging to it.

And now, a nonexhaustive list of open questions.

Question. Characterize numbers belonging to Class 3, 4, and 5.

Question. Are there rational numbers in Classes 3, 4, and 5, or only in some of them?

Question. Do Perron numbers not belonging to Classes 1 and 2 belong to some particular class(es) among the others?

Question. Specifically, to what classes do π , e , $\frac{3}{2}$ belong?

Question. Prove (if true) that Class 5 has full Lebesgue measure.

Question. Are there transcendental numbers in Classes 3, 4 and 5? (this question is partly answered in the Appendix).

4.2. Periodic points under f_β

The last question is pure number theory, but it might indirectly concern symbolic dynamics. Schmidt [18] has proved that for all rational numbers to have an ultimately periodic β -expansion, β must be either Pisot or Salem.

Question. (Schmidt [18]). Is the converse true?

There are also other related questions.

Appendix

A transcendent number for which S_β has the specification property (J.-P. Allouche)

Consider the sequence $s = (s_{n+1})$, $n \in \mathbb{N}$, where s_n is the sum (mod 2) of digits of n in base 2. It has the following properties:

(1) $\forall k > 0, \sigma^k(s) < s$. Hence it is the expansion of 1 for a unique β (Proposition 2.4). Proof can be found in [1].

(2) The associated β is transcendental (else the value $\sum s_n \beta^{-n}$ would be, which is obviously false: cf. [15, p. 363]).

(3) There never appear in s more than two consecutive 0's (cf. [20]).

All *q-mirror* numbers introduced in [1] have the same properties. They form a denumerable set.

A transcendental number for which S_β is not specified (C. Mauduit)

It is given by Example 4.8. The associated β is transcendental: this results from the "almost completely proved" theorem of Loxton and Van der Poorten: any number for which the expansion of 1 is generated by an automaton and not ultimately periodic is transcendental.

(A. Bertrand-Mathis). It is easy to construct a sequence s with $\sigma^k(s) < s$ containing steadily increasing strings of 0's so that the associated β must be a Liouville number, hence transcendental. Of course, the number of consecutive 0's cannot be bounded.

Bibliography

On β -expansions and subshifts S_β

Both notions were introduced in [17] and developed in [16]. Papers [19, 13, 18, 2-5] are contributions about probabilistic and topological features of the question. There is also something about it in [7].

On symbolic dynamics

Basic notions are to be found in [11]. Some complements on sofic systems are in [21], on coded and synchronizing systems and the relationship between languages and subshifts in [8]. In [4] it is proven that all specified systems are synchronizing. [14] shows the relationship between Perron numbers and sofic systems. The correlation between $d(1, \beta)$ and ergodic properties of S_β is explored in [4], in [7] exhaustive systems are studied.

About languages

See [12] about languages in general, [10] about codes.

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